

Spherically Symmetric Cosmology: Resource Paper

A. A. Coley, W. C. Lim and G. Leon

March 6, 2008

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5 and Universidad Central de Las Villas, Santa Clara, CP 54830, Cuba.

E-mail: aac@mathstat.dal.ca, wlim@princeton.edu, genly@uclv.edu.cu

Abstract

We use the 1+3 frame formalism to write down the evolution equations for spherically symmetric models as a well-posed system of first order PDEs in two variables, suitable for numerical and qualitative analysis.

1 Introduction

We shall use the 1+3 frame formalism [1, 2] to write down the evolution equations for spherically symmetric models as a well-posed system of first order PDEs in 2 variables. The formalism is particularly well-suited for studying perfect fluid spherically symmetric models [3], and especially for numerical and qualitative analysis, and is useful in various applications, such as structure formation in the spherically symmetric dust Lemaître-Tolman-Bondi model. This preprint is intended as a resource paper for researchers working in this field.

2 Spherically symmetric models

The metric is:¹

$$ds^2 = -N^2 dt^2 + (e_1^1)^{-2} dx^2 + (e_2^2)^{-2} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (1)$$

The Killing vector fields (KVF) are given by [4]:

$$\partial_\varphi, \quad \cos \varphi \partial_\vartheta - \sin \varphi \cot \vartheta \partial_\varphi, \quad \sin \varphi \partial_\vartheta + \cos \varphi \cot \vartheta \partial_\varphi. \quad (2)$$

The frame vectors in coordinate form are:

$$\mathbf{e}_0 = N^{-1} \partial_t, \quad \mathbf{e}_1 = e_1^1 \partial_x, \quad \mathbf{e}_2 = e_2^2 \partial_\vartheta, \quad \mathbf{e}_3 = e_3^3 \partial_\varphi, \quad (3)$$

where $e_3^3 = e_2^2 / \sin \vartheta$. N , e_1^1 and e_2^2 are functions of t and x .²

This leads to the following restrictions on the kinematic variables:

$$\sigma_{\alpha\beta} = \text{diag}(-2\sigma_+, \sigma_+, \sigma_+), \quad \omega_{\alpha\beta} = 0, \quad \dot{u}_\alpha = (\dot{u}_1, 0, 0), \quad (4)$$

where

$$\dot{u}_1 = \mathbf{e}_1 \ln N; \quad (5)$$

on the spatial commutation functions:

$$a_\alpha = (a_1, a_2, 0), \quad n_{\alpha\beta} = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & 0 & 0 \\ n_{13} & 0 & 0 \end{pmatrix}, \quad (6)$$

where³

$$a_1 = \mathbf{e}_1 \ln e_2^2, \quad a_2 = n_{13} = -\frac{1}{2} e_2^2 \cot \vartheta; \quad (7)$$

and on the matter components:

$$q_\alpha = (q_1, 0, 0), \quad \pi_{\alpha\beta} = \text{diag}(-2\pi_+, \pi_+, \pi_+). \quad (8)$$

¹We use x instead of r because r is used to denote the spatial derivative of H .

²Note that the frame vectors \mathbf{e}_2 and \mathbf{e}_3 tangent to the spheres are not group-invariant – the commutators $[\mathbf{e}_2, \partial_\varphi]$ and $[\mathbf{e}_3, \partial_\varphi]$ are zero, but not with the other two Killing vectors. The frame vectors \mathbf{e}_0 and \mathbf{e}_1 orthogonal to the spheres are group-invariant.

³The dependence of a_2 and n_{13} on ϑ is due to the fact that the chosen orthonormal frame is not group-invariant. However, this is not a concern, since the ϑ dependence will be hidden.

The frame rotation $\Omega_{\alpha\beta}$ is also zero.

Furthermore, n_{13} only appears in the equations together with $\mathbf{e}_2 n_{13}$ in the form of the Gauss curvature of the spheres

$${}^2K := 2(\mathbf{e}_2 - 2n_{13})n_{13}, \quad (9)$$

which simplifies to

$${}^2K = (e_2^2)^2. \quad (10)$$

Thus the dependence on ϑ is hidden in the equations. We will also use 2K in place of e_2^2 .

The spatial curvatures also simplify to:

$${}^3S_{\alpha\beta} = \text{diag}(-2{}^3S_+, {}^3S_+, {}^3S_+), \quad (11)$$

with 3R and ${}^3S_+$ given by:

$${}^3R = 4\mathbf{e}_1 a_1 - 6a_1^2 + 2{}^2K \quad (12)$$

$${}^3S_+ = -\frac{1}{3}\mathbf{e}_1 a_1 + \frac{1}{3}{}^2K. \quad (13)$$

The Weyl curvature components simplify to:

$$E_{\alpha\beta} = \text{diag}(-2E_+, E_+, E_+), \quad H_{\alpha\beta} = 0, \quad (14)$$

with E_+ given by

$$E_+ = H\sigma_+ + \sigma_+^2 + {}^3S_+ - \frac{1}{2}\pi_+. \quad (15)$$

To simplify notation, we will write

$${}^2K, \dot{u}_1, a_1$$

as

$$K, \dot{u}, a.$$

To summarize, the essential variables are

$$N, e_1^1, K, H, \sigma_+, a, \mu, q_1, p, \pi_+, \quad (16)$$

and the auxiliary variables are

$${}^3R, {}^3S_+, \dot{u}. \quad (17)$$

So far, there are no evolution equations for N , p and π_+ , and they need to be specified by a temporal gauge (for N), and by a fluid model (for p and π_+).

The evolution equations are now:⁴

$$\mathbf{e}_0 e_1^1 = (-H + 2\sigma_+) e_1^1 \quad (18)$$

$$\mathbf{e}_0 K = -2(H + \sigma_+) K \quad (19)$$

$$\mathbf{e}_0 H = -H^2 - 2\sigma_+^2 + \frac{1}{3}(\mathbf{e}_1 + \dot{u} - 2a)\dot{u} - \frac{1}{6}(\mu + 3p) + \frac{1}{3}\Lambda \quad (20)$$

$$\mathbf{e}_0 \sigma_+ = -3H\sigma_+ - \frac{1}{3}(\mathbf{e}_1 + \dot{u} + a)\dot{u} - {}^3S_+ + \pi_+ \quad (21)$$

$$\mathbf{e}_0 a = (-H + 2\sigma_+)a - (\mathbf{e}_1 + \dot{u})(H + \sigma_+) \quad (22)$$

$$\mathbf{e}_0 \mu = -3H(\mu + p) - (\mathbf{e}_1 + 2\dot{u} - 2a)q_1 - 6\sigma_+ \pi_+ \quad (23)$$

$$\mathbf{e}_0 q_1 = (-4H + 2\sigma_+)q_1 - \mathbf{e}_1 p - (\mu + p)\dot{u} + 2(\mathbf{e}_1 + \dot{u} - 3a)\pi_+. \quad (24)$$

The constraint equations are the Gauss and Codazzi constraints, and the definition of a :

$$0 = 3H^2 + \frac{1}{2}{}^3R - 3\sigma_+^2 - \mu - \Lambda \quad (25)$$

$$0 = -2\mathbf{e}_1(H + \sigma_+) + 6a\sigma_+ + q_1 \quad (26)$$

$$0 = (\mathbf{e}_1 - 2a)K, \quad (27)$$

where the spatial curvatures are given by

$${}^3R = 4\mathbf{e}_1 a - 6a^2 + 2K \quad (28)$$

$${}^3S_+ = -\frac{1}{3}\mathbf{e}_1 a + \frac{1}{3}K. \quad (29)$$

3 The matter and gauge

There are various choices for the matter.

3.1 Perfect fluid

A perfect fluid is defined by [2]

$$T_{ab} = \hat{\mu}u_a u_b + \hat{p}(g_{ab} + u_a u_b). \quad (30)$$

⁴We include a non-negative Λ .

with \hat{p} to be specified. In general, the 4-velocity vector \mathbf{u} of the perfect fluid is not aligned with the vector \mathbf{e}_0 of a chosen temporal gauge. In spherically symmetric models, \mathbf{u} is allowed to be of the form

$$\mathbf{u} = \Gamma(\mathbf{e}_0 + v\mathbf{e}_1), \quad \Gamma = (1 - v^2)^{-\frac{1}{2}}. \quad (31)$$

We choose a linear equation of state for the perfect fluid:

$$\hat{p} = (\gamma - 1)\hat{\mu}, \quad (32)$$

where γ is a constant satisfying $1 \leq \gamma < 2$. Then we obtain for the tilted fluid:

$$\mu = \frac{G_+}{1 - v^2}\hat{\mu} \quad (33)$$

$$p = \frac{(\gamma - 1)(1 - v^2) + \frac{1}{3}\gamma v^2}{G_+}\mu \quad (34)$$

$$q_1 = \frac{\gamma\mu}{G_+}v \quad (35)$$

$$\pi_+ = -\frac{1}{3}\frac{\gamma\mu}{G_+}v^2, \quad (36)$$

where $G_{\pm} = 1 \pm (\gamma - 1)v^2$. Thus p , q_1 and π_+ are given in terms of μ and v . These are then substituted into the evolution and constraint equations.

The evolution equations for μ and q_1 now give (in terms of μ and v)

$$\mathbf{e}_0\mu = -\frac{\gamma v}{G_+}\mathbf{e}_1\mu - \frac{\gamma G_-}{G_+^2}\mu\mathbf{e}_1v - \frac{\gamma}{G_+}\mu[(3 + v^2)H + 2v(\dot{u} - a) - 2v^2\sigma_+] \quad (37)$$

$$\begin{aligned} \mathbf{e}_0v = & -\frac{(\gamma - 1)(1 - v^2)^2}{\gamma G_- \mu}\mathbf{e}_1\mu + \frac{[(3\gamma - 4) - (\gamma - 1)(4 - \gamma)v^2]v}{G_+ G_-}\mathbf{e}_1v \\ & - \frac{(1 - v^2)}{G_-}[-(3\gamma - 4)vH - 2v\sigma_+ + G_- \dot{u} + 2(\gamma - 1)v^2a]. \end{aligned} \quad (38)$$

3.2 Scalar fields and anisotropic fluid

The total energy-momentum tensor of a non-interacting scalar field ϕ with a self-interaction potential $V(\phi)$ is

$$T_{ab}^{sf} = \phi_{;a}\phi_{;b} - g_{ab}(\frac{1}{2}\phi_{;c}\phi^{;c} + V(\phi)), \quad (39)$$

where $\phi = \phi(t, x)$. In particular, exponential potentials have been the subject of much interest and arise naturally from theories of gravity such as scalar-tensor theories or string theory [5]. Spherically symmetric scalar field models have been studied in [6].

A spherically symmetric model can also admit an anisotropic fluid matter source, in which the energy momentum tensor has energy density μ , a pressure $p_{||}$ parallel to the radial unit normal and a perpendicular pressure p_{\perp} . Fluids with an anisotropic pressure have been studied in the cosmological context for a number of reasons [7]. An energy-momentum tensor of this form formally arises if the source consists of two perfect fluids with distinct four-velocities, a heat conducting viscous fluid and a perfect fluid and a magnetic field; in addition, a cosmic string and a global monopole are of the form of an anisotropic fluid. Most importantly, perhaps, a contribution in the form of an anisotropic fluid arises when averaging the Einstein equation to obtain the averaged field equations in spherically symmetric geometries [8].

3.3 Temporal gauge

The common temporal gauges used in spherically symmetric cosmological models are the *synchronous gauge* and the *separable area gauge*. The synchronous gauge is useful when used with a dust perfect fluid ($\gamma = 1$) because the dust perfect fluid has zero acceleration ($\dot{u} = 0$). This gives the Lemaître-Tolman-Bondi models. It may also be useful when used with a non-dust tilted perfect fluid. The un-normalized system is well-posed when the Gauss and Codazzi constraints are used to eliminate the spatial derivatives. H -normalization preserves well-posedness.

The separable area gauge has a special case ($a = 0$), called the *timelike area gauge*. The un-normalized system is well-posed when $\mathbf{e}_0(H + \sigma_+)$ is used, and the Gauss constraint is solved for $H - \sigma_+$. $(H + \sigma_+)$ -normalization preserves well-posedness, while H -normalization does not.

4 Special cases with extra Killing vectors

Spherically symmetric models with more than 3 KVF are either spatially homogeneous or static. Let us discuss the spatially homogeneous cosmological models. Spatially homogeneous spherically symmetric models consist of

two disjoint sets of models: the Kantowski-Sachs models and the Friedmann-Lemaître-Robertson-Walker (FLRW) models. Static and self-similar spherically symmetric models have been studied in [6, 9, 3].

4.1 The Kantowski-Sachs models

The spatially homogeneous spherically symmetric models (that has 4 Killing vectors, the fourth being ∂_x) are the so-called Kantowski-Sachs models [4]. The metric (1) simplifies to

$$ds^2 = -N(t)^2 dt^2 + (e_1^1(t))^{-2} dx^2 + (e_2^2(t))^{-2} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2); \quad (40)$$

i.e., N , e_1^1 and e_2^2 are now independent of x .

The spatial derivative terms $\mathbf{e}_1(\cdot)$ vanish and as a result $a = 0 = \dot{u}$. Since $\dot{u} = 0$, the temporal gauge is synchronous and we can set N to any positive function of t .

The Codazzi constraint restricts the source by

$$q_1 = 0. \quad (41)$$

p and π_+ are still unspecified.

The evolution equations for Kantowski-Sachs models with unspecified source are:

$$\mathbf{e}_0 e_1^1 = (-H + 2\sigma_+) e_1^1 \quad (42)$$

$$\mathbf{e}_0 K = -2(H + \sigma_+) K \quad (43)$$

$$\mathbf{e}_0 H = -H^2 - 2\sigma_+^2 - \frac{1}{6}(\mu + 3p) + \frac{1}{3}\Lambda \quad (44)$$

$$\mathbf{e}_0 \sigma_+ = -3H\sigma_+ - \frac{1}{3}K + \pi_+ \quad (45)$$

$$\mathbf{e}_0 \mu = -3H(\mu + p) - 6\sigma_+ \pi_+ \quad (46)$$

The remaining constraint equation is the Gauss constraint:

$$0 = 3H^2 + K - 3\sigma_+^2 - \mu - \Lambda. \quad (47)$$

The spatial curvatures are given by

$${}^3R = 2K \quad (48)$$

$${}^3S_+ = \frac{1}{3}K. \quad (49)$$

4.2 The FLRW models

Spatially homogeneous spherically symmetric models, that are not Kantowski-Sachs, are the Friedmann-Lemaître-Robertson-Walker (FLRW) models (with or without Λ). The source must be of the form of a comoving perfect fluid (or vacuum).

The metric has the form

$$ds^2 = -N(t)^2 dt^2 + \ell^2(t) dx^2 + \ell^2(t) f^2(x) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (50)$$

with

$$f(x) = \sin x, \quad x, \quad \sinh x, \quad (51)$$

for closed, flat, and open FLRW models respectively. The frame coefficients are given by $e_1^1 = \ell^{-1}(t)$ and $e_2^2 = \ell^{-1}(t) f^{-1}(x)$. Then $\sigma_+ = \frac{1}{3} \mathbf{e}_0 \ln(e_1^1/e_2^2)$ vanishes. $N = N(t)$ implies that $\dot{u} = 0$; i.e., the temporal gauge is synchronous, and we can set N to any positive function of t . The Hubble scalar $H = \mathbf{e}_0 \ln \ell(t)$ is also a function of t .⁵

For the spatial curvatures, ${}^3S_+$ does vanish because (51) implies $\mathbf{e}_1 a = K$,⁶ while 3R simplifies to

$${}^3R = \frac{6k}{\ell^2}, \quad k = 1, 0, -1, \quad (52)$$

for closed, flat, and open FLRW respectively.

The evolution equation for σ_+ and the Codazzi constraint then imply that $\pi_+ = 0 = q_1$; i.e., the source is a comoving perfect fluid, with unspecified pressure p .

The evolution equations simplify to:

$$\mathbf{e}_0 \ell = H \ell \quad (53)$$

$$\mathbf{e}_0 H = -H^2 - \frac{1}{6}(\mu + 3p) + \frac{1}{3}\Lambda \quad (54)$$

$$\mathbf{e}_0 \mu = -3H(\mu + p) \quad (55)$$

The Gauss constraint simplifies to

$$0 = 3H^2 + \frac{3k}{\ell^2} - \mu - \Lambda, \quad k = 1, 0, -1. \quad (56)$$

⁵We shall not list the KVF's as they are complicated in spherically symmetric coordinates and not needed here.

⁶That $\mathbf{e}_1 a$ does not vanish is consistent with the frame vector \mathbf{e}_1 not being group-invariant.

Note that μ and p also depend on t only, and that p is not specified yet.

The vacuum cases are the de Sitter model ($\Lambda > 0$, $k = 0$), the model with $\Lambda > 0$, $k = 1$, the model with $\Lambda > 0$, $k = -1$, the Milne model ($\Lambda = 0$, $k = -1$), and the Minkowski spacetime ($\Lambda = 0$, $k = 0$), which is also static. The model with $\Lambda > 0$, $k = 1$ is past asymptotic to the de Sitter model with negative H and is future asymptotic to the de Sitter model with positive H . The model with $\Lambda > 0$, $k = -1$ (and positive H) is past asymptotic to the Milne model and is future asymptotic to the de Sitter model with positive H .

5 Synchronous gauge, tilted perfect fluid

We shall investigate perfect fluid models with linear equation of state using the synchronous gauge. We shall simplify the equations step-by-step, by choosing the synchronous gauge, eliminating spatial derivatives, and specifying the perfect fluid.

The equations in synchronous gauge ($\dot{u} = 0$) are:

$$\mathbf{e}_0 e_1^1 = (-H + 2\sigma_+)e_1^1 \quad (57)$$

$$\mathbf{e}_0 K = -2(H + \sigma_+)K \quad (58)$$

$$\mathbf{e}_0 H = -H^2 - 2\sigma_+^2 - \frac{1}{6}(\mu + 3p) + \frac{1}{3}\Lambda \quad (59)$$

$$\mathbf{e}_0 \sigma_+ = -3H\sigma_+ - {}^3S_+ + \pi_+ \quad (60)$$

$$\mathbf{e}_0 a = (-H + 2\sigma_+)a - \mathbf{e}_1(H + \sigma_+) \quad (61)$$

$$\mathbf{e}_0 \mu = -3H(\mu + p) - (\mathbf{e}_1 - 2a)q_1 - 6\sigma_+\pi_+ \quad (62)$$

$$\mathbf{e}_0 q_1 = (-4H + 2\sigma_+)q_1 - \mathbf{e}_1 p + 2(\mathbf{e}_1 - 3a)\pi_+. \quad (63)$$

The constraint equations are:

$$0 = 3H^2 + \frac{1}{2}{}^3R - 3\sigma_+^2 - \mu - \Lambda \quad (64)$$

$$0 = -2\mathbf{e}_1(H + \sigma_+) + 6a\sigma_+ + q_1 \quad (65)$$

$$0 = (\mathbf{e}_1 - 2a)K. \quad (66)$$

where the spatial curvatures are given by

$${}^3R = 4\mathbf{e}_1 a - 6a^2 + 2K \quad (67)$$

$${}^3S_+ = -\frac{1}{3}\mathbf{e}_1 a + \frac{1}{3}K. \quad (68)$$

The evolution equations (60) and (61) contain spatial derivative terms, but these can be replaced using the constraints (64) and (65):

$$\mathbf{e}_1 a = -\frac{3}{2}H^2 + \frac{3}{2}a^2 - \frac{1}{2}K + \frac{3}{2}\sigma_+^2 + \frac{1}{2}\mu + \frac{1}{2}\Lambda \quad (69)$$

$$\mathbf{e}_1(H + \sigma_+) = 3a\sigma_+ + \frac{1}{2}q_1. \quad (70)$$

As a result, equations (60) and (61) now read:

$$\mathbf{e}_0\sigma_+ = -3H\sigma_+ - \frac{1}{2}H^2 + \frac{1}{2}a^2 - \frac{1}{2}K + \frac{1}{2}\sigma_+^2 + \frac{1}{6}\mu + \frac{1}{6}\Lambda + \pi_+ \quad (71)$$

$$\mathbf{e}_0 a = -(H + \sigma_+)a - \frac{1}{2}q_1. \quad (72)$$

The benefit here is that the evolution equations for the geometric part are now free of spatial derivative terms.

The spatial curvatures are given by

$${}^3R = -6H^2 + 6\sigma_+^2 + 2\mu + 2\Lambda \quad (73)$$

$${}^3S_+ = \frac{1}{2}H^2 - \frac{1}{2}a^2 + \frac{1}{2}K - \frac{1}{2}\sigma_+^2 - \frac{1}{6}\mu - \frac{1}{6}\Lambda. \quad (74)$$

Lastly, we specify the perfect fluid with linear equation of state. From equations (32)–(38) and the above equations, the final form of the system is:

$$\mathbf{e}_0 e_1^1 = (-H + 2\sigma_+)e_1^1 \quad (75)$$

$$\mathbf{e}_0 K = -2(H + \sigma_+)K \quad (76)$$

$$\mathbf{e}_0 H = -H^2 - 2\sigma_+^2 - \frac{(3\gamma - 2 + (2 - \gamma)v^2)}{6G_+}\mu + \frac{1}{3}\Lambda \quad (77)$$

$$\mathbf{e}_0\sigma_+ = -3H\sigma_+ - \frac{1}{2}H^2 + \frac{1}{2}a^2 - \frac{1}{2}K + \frac{1}{2}\sigma_+^2 + \frac{(1 - (\gamma + 1)v^2)}{6G_+}\mu + \frac{1}{6}\Lambda \quad (78)$$

$$\mathbf{e}_0 a = -(H + \sigma_+)a - \frac{\gamma v}{2G_+}\mu \quad (79)$$

$$\begin{aligned} \mathbf{e}_0\mu + \frac{\gamma v}{G_+}\mathbf{e}_1\mu + \frac{\gamma G_-}{G_+^2}\mu\mathbf{e}_1v \\ = -\frac{\gamma}{G_+}\mu [(3 + v^2)H - 2va - 2v^2\sigma_+] \end{aligned} \quad (80)$$

$$\begin{aligned} \mathbf{e}_0 v + \frac{(\gamma - 1)(1 - v^2)^2}{\gamma G_- \mu}\mathbf{e}_1\mu - \frac{[(3\gamma - 4) - (\gamma - 1)(4 - \gamma)v^2]v}{G_+ G_-}\mathbf{e}_1 v \\ = -\frac{(1 - v^2)}{G_-} [-(3\gamma - 4)H - 2\sigma_+ + 2(\gamma - 1)va] v, \end{aligned} \quad (81)$$

where $G_{\pm} = 1 \pm (\gamma - 1)v^2$. The constraints are:

$$\mathbf{e}_1 a = -\frac{3}{2}H^2 + \frac{3}{2}a^2 - \frac{1}{2}K + \frac{3}{2}\sigma_+^2 + \frac{1}{2}\mu + \frac{1}{2}\Lambda \quad (82)$$

$$\mathbf{e}_1(H + \sigma_+) = 3a\sigma_+ + \frac{\gamma v}{2G_+}\mu \quad (83)$$

$$\mathbf{e}_1 K = 2aK. \quad (84)$$

The spatial curvatures are given by

$${}^3R = -6H^2 + 6\sigma_+^2 + 2\mu + 2\Lambda \quad (85)$$

$${}^3S_+ = \frac{1}{2}H^2 - \frac{1}{2}a^2 + \frac{1}{2}K - \frac{1}{2}\sigma_+^2 - \frac{1}{6}\mu - \frac{1}{6}\Lambda. \quad (86)$$

5.1 Well-posedness

We now show that the system is well-posed for $\gamma \geq 1$.

The coefficient matrix for the spatial derivative terms is: ⁷

$$\begin{pmatrix} \frac{\gamma v}{G_+} & \frac{\gamma G_-}{G_+^2}\mu \\ \frac{(\gamma - 1)(1 - v^2)^2}{\gamma G_- \mu} & -\frac{[(3\gamma - 4) - (\gamma - 1)(4 - \gamma)v^2]v}{G_+ G_-} \end{pmatrix}. \quad (87)$$

Its eigenvalues are

$$\frac{(2 - \gamma)v \pm \sqrt{\gamma - 1}(1 - v^2)}{G_-}, \quad (88)$$

with corresponding eigenvectors (for example)

$$\begin{pmatrix} 1 \\ \frac{\sqrt{\gamma - 1}(1 - v^2)G_+}{\mu\gamma(1 + \sqrt{\gamma - 1}v)^2} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\frac{\sqrt{\gamma - 1}(1 - v^2)G_+}{\mu\gamma(1 - \sqrt{\gamma - 1}v)^2} \end{pmatrix}. \quad (89)$$

The matrix is diagonalizable for $\gamma > 1$, with $c_s = \sqrt{\gamma - 1}$ being the speed of sound in the perfect fluid. The system (75)–(81) is thus well-posed for $\gamma > 1$. For $\gamma < 1$ the system is elliptic and not well-posed.

⁷Strictly speaking, we should also include the factor e_1^1/N in the matrix, but the result on well-posedness is the same.

6 Irrotational dust (Lemaître-Tolman-Bondi model)

The Lemaître-Tolman-Bondi (LTB) model [10, 7] is the spherically symmetric dust solution of the Einstein equations which can be regarded as a generalization of the FLRW universe. LTB metrics with dust source and a comoving and geodesic 4-velocity constitute a well known class of exact solutions of Einstein's field equations [4, 7].

For the dust case $\gamma = 1$ with zero vorticity, we can use the freedom within the synchronous gauge to set $v = 0$, so that the synchronous frame is comoving with the perfect fluid and we obtain:

$$\mathbf{e}_0 e_1^1 = (-H + 2\sigma_+) e_1^1 \quad (90)$$

$$\mathbf{e}_0 K = -2(H + \sigma_+) K \quad (91)$$

$$\mathbf{e}_0 H = -H^2 - 2\sigma_+^2 - \frac{1}{6}\mu + \frac{1}{3}\Lambda \quad (92)$$

$$\mathbf{e}_0 \sigma_+ = -3H\sigma_+ - \frac{1}{2}H^2 + \frac{1}{2}a^2 - \frac{1}{2}K + \frac{1}{2}\sigma_+^2 + \frac{1}{6}\mu + \frac{1}{6}\Lambda \quad (93)$$

$$\mathbf{e}_0 a = -(H + \sigma_+) a \quad (94)$$

$$\mathbf{e}_0 \mu = -3H\mu. \quad (95)$$

Notice that the system is completely free of spatial derivatives, and is thus well-posed.

The constraints are:

$$\mathbf{e}_1 a = -\frac{3}{2}H^2 + \frac{3}{2}a^2 - \frac{1}{2}K + \frac{3}{2}\sigma_+^2 + \frac{1}{2}\mu + \frac{1}{2}\Lambda \quad (96)$$

$$\mathbf{e}_1 (H + \sigma_+) = 3a\sigma_+ \quad (97)$$

$$\mathbf{e}_1 K = 2aK. \quad (98)$$

The spatial curvatures are given by

$${}^3R = -6H^2 + 6\sigma_+^2 + 2\mu + 2\Lambda \quad (99)$$

$${}^3S_+ = \frac{1}{2}H^2 - \frac{1}{2}a^2 + \frac{1}{2}K - \frac{1}{2}\sigma_+^2 - \frac{1}{6}\mu - \frac{1}{6}\Lambda. \quad (100)$$

Further simplifications with a and K are possible.

A suitable normalization factor is $\beta = H + \sigma_+$, introduced for G_2 models in [2, 1], and used for the LTB model in [11]. With this normalization, it can be shown that at late times the LTB solutions that are ever-expanding will

tend to the isotropic and homogeneous Milne solution, with the following rates:

$$\beta \sim e^{-\tau}, \quad \frac{a^2 - K}{\beta^2} - 1 \sim e^{-\tau}, \quad (101)$$

$$\frac{\sigma_+}{\beta} \sim \tau e^{-\tau}, \quad \frac{\mu}{3\beta^2} \sim e^{-\tau}. \quad (102)$$

That is, the rates are the same for all dust observers, although the multiplicative “constants” depend on the radius. This dependency reveals itself in the leading order of ratios of variables such as the density contrast.

6.1 Structure formation

Structure formation in the LTB model has been studied in [12]. More recently, the LTB inhomogeneous dust solutions have been examined numerically and qualitatively as a 3-dimensional dynamical system, in terms of an average density parameter, $\langle \Omega \rangle$ (which behaves dynamically like the usual Ω in FLRW dust spacetimes), and a shear parameter and a density contrast function which convey the effects of inhomogeneities [11]. The evolution equations for the averaged variables are formally identical to those of an equivalent FLRW cosmology, and are an alternative set of evolution equations to those presented above. In particular, the phase space evolution of structure formation scenario was examined in [11].

References

- [1] H. van Elst and C. Ugla, *Class. Quant. Grav.* **14** 2673 (1997).
- [2] J. Wainwright and G. F. R. Ellis (editors), *Dynamical Systems in Cosmology* (Cambridge University Press, 1997).
- [3] B.J. Carr and A.A. Coley, *Class. Q. Grav.* **16**, R31 (1999) [gr-qc/9806048]; B.J. Carr and A.A. Coley, *Phys. Rev. D.* **62**, 044023 (2000) [gr-qc/9901050]; B. J. Carr, A. A. Coley, M. Goliath, U. S. Nilsson, C. Ugla, *Class. Quant. Grav.* **18** 303 (2001); M. Goliath, U. S. Nilsson, C. Ugla, *Class. Quant. Grav.* **15** 167 & 2841 (1998).

- [4] H. Stephani, D. Kramer, M. A. H. MacCallum, C. A. Hoenselaers, E. Herlt, *Exact solutions of Einstein's field equations, second edition* (Cambridge University Press, Cambridge, 2003).
- [5] A. A. Coley, *Dynamical Systems and Cosmology* (Kluwer, Academic Publishers, 2003).
- [6] G. Leon, A. A. Coley and W. C. Lim; preprint.
- [7] A. Krasinski, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, 1997).
- [8] A. A. Coley and N. Pelavas, Phys. Rev. D. **74** 087301 (2006) [gr-qc/0607079]; A. A. Coley and N. Pelavas, Phys. Rev. D **75** 043506 (2006) [gr-qc/0607079]; A.A. Coley, N. Pelavas and R.M. Zalaletdinov, Phys. Rev. Letts. **595**, 115102 (2005) [gr-qc/0504115]
- [9] U. S. Nilsson, C. Uggla, Annals Phys. **286** 278 & 292 (2001).
- [10] A. G. Lemaitre, Gen. Rel. Grav. **29** 641 (1997) (reprint); R. C. Tolman, Gen. Rel. Grav. **29** 935 (1997) (reprint); H. Bondi, Mon. Not. R. Astron. Soc. **107** 410 (1947).
- [11] R. A. Sussman, Class. Quantum Grav., **25**, (2008) 015012; see also arXiv:0801.3324.
- [12] A. Krasinski and C. Hellaby, Phys. Rev. D **65** 023501 (2002) and Phys. Rev. D **69** 023502 (2004).